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Mean velocity and longitudinal dispersion of heavy particles in turbulent open-channel flow

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This paper deals with the motion of a heavy particle in a turbulent flow in an open channel with a smooth bottom. For the case when the particle stays in suspension in the main body of the flow almost all the time, (*a*) the probability density function of the projection on a cross-sectional plane of the particle position at any instant, and (*b*) the mean velocity and longitudinal dispersion coefficient of particles are determined analytically by employing the Eulerian formulation and applying the Aris moment transformations. It is found that the mean particle velocity decreases and the longitudinal dispersion coefficient of particles increases with the fall velocity.

1. Introduction

When a single heavy particle whose size is relatively small compared with the depth is released into a turbulent open-channel flow, it moves under the combined action of the mean shearing motion of the fluid, turbulence and gravity. Depending on both the flow conditions close to the bottom of the channel and the particle characteristics, the particle either will continue on its way mostly in the main body of the flow, or will drop out of the flow owing to gravity. In the case where the conditions allow the particle to travel in suspension, the quantities needed to describe the motion in the Lagrangian sense appear to be statistical moments of the particle displacement, since the particle motion in this particular case looks like that of a fluid particle due to turbulent diffusion. In practice, this so-called sediment suspension is what occurs in fluvial streams where silt and sediment particles the size of fine sand are transported mostly in the main body of the flow. Prediction of the first two statistical moments of the particle displacement in the flow direction, the mean and variance, would mean that one could calculate the mean velocity and the longitudinal dispersion coefficient of sediment particles.

Batchelor, Binnie & Phillips (1955) studied the mean velocity and the longitudinal dispersion of particles in a circular pipe and showed that the time-averaged axial component of velocity of a fluid particle is equal to the discharge velocity. Binnie & Phillips (1958) extended this work to cover slightly heavy and buoyant particles. Barnard & Binnie (1963) extended this study one step further to include heavy particles.

Elder (1959) applied the analysis developed by Taylor (1954) to describe the longitudinal dispersion of discrete particles, both of zero and of finite buoyancy. The relevance of Elder's prediction to the present study will be discussed in more detail in this paragraph. Using the expression [see equation (A 6)] for the probability density of the particle position in the vertical and assuming a parabolic velocity distribution in the channel, Elder calculated the mean velocity of heavy particles and then predicted the dispersion coefficient. In predicting the dispersion coefficient he used an expression (Elder 1959, formula 9) derived from the equation of conservation of mass which, in that case, did not contain a term characterizing the settling of particles due to gravity. Because an artificial velocity distribution was used in the calculation, and the main equation in the derivation of the expression for the dispersion coefficient should have had a fall-velocity term, Elder's estimate differs from Sayre's (1968) numerical calculation and the author's prediction presented in this paper.

Batchelor (1965), in his review on the motion of small particles in turbulent flow, discussed the systematic effects of inertia difference or more important, the action of gravity on the motion of a heavy particle. In particular, he examined the equation representing the balance between transport due to gravity and turbulent transport near the bottom of an open-channel flow. From this there can be obtained a criterion for whether the particles stay in suspension or not (see § 2.2).

The exchange of particles between the bed and the flow in a fluvial stream has not been explained physically and no quantitative criterion has been established so far. However, to be able to analyse the dispersion process analytically it is necessary in this case to work with a hypothetical model which permits the exchange of particles in a certain manner. Sayre (1968, 1969) formulated the dispersion process of sediment particles in an open-channel flow in the Eulerian sense, permitting the exchange of particles by introducing a so-called bed absorbency coefficient and an entrainment-rate coefficient. Using the Aris moment transformations, he obtained zeroth, first, second and third moments of the concentration numerically with the aid of a digital computer. In an earlier study the author examined the specific case in which the ratio of the fall velocity of particles to the shear velocity is small compared with unity, which probably implies that the particles are all transported in suspension and the mean particle velocity is approximately equal to that of the flow (Sumer 1971). Applying the method used by Taylor (1954) he obtained an expression for the longitudinal dispersion coefficient.

Particle motion in a turbulent channel flow, particularly close to the bottom, presents quite a complex problem, especially when particle exchange between the bottom and the flow is involved. To eliminate the difficulties which arise owing to the nature of the assumption about the bottom, we shall suppose the bottom of the channel to be smooth.† The specific goal of this study is to determine the mean velocity and the longitudinal dispersion coefficient of heavy particles; these are the most important quantities needed to describe the particle motion, when conditions are such that the particle travels in suspension almost all the time.

† This assumption is clearly not connected with the process of dispersion of particles in an open-channel flow when the particles are all transported in suspension (see Batchelor 1965).

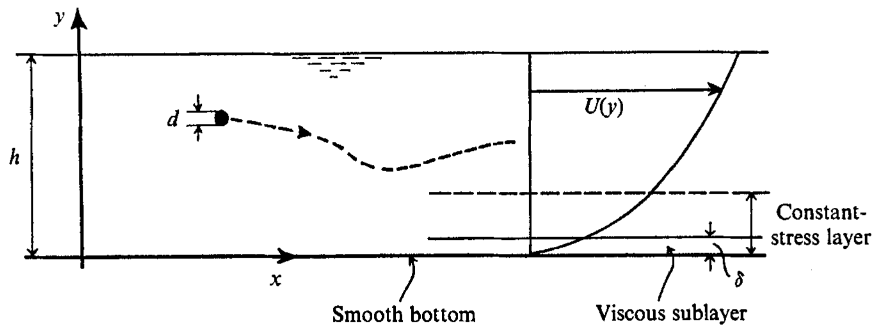

FIGURE 1. Definition sketch.

2. General considerations

2.1. *Statement of the problem*

The flow is a two-dimensional fully developed shear flow in an open channel with a smooth bottom. The flow depth is h and the x and y axes are chosen so that x is in the flow direction and y in the upward direction (figure 1). The process is considered to be independent of the lateral position. The particle is such that its size d is small compared with the flow depth h. The terminal velocity of the particle in quiescent fluid is denoted by w. The main problem in this study is to describe the motion of a particle when it is released from a point in the channel and travels under the combined action of the mean shearing motion, turbulence and gravity.

2.2. *Particle motion close to the bottom*

As has been reported by many researchers, observations made in a laboratory channel show that for certain flow speeds a small heavy particle sits motionless on the bottom. For some higher speeds it appears to move steadily along the bottom, while for yet higher speeds it may be lifted into the body of the flow.

Batchelor (1965) examined the constant-stress layer when considering particle transport in channel flows. His argument will be given briefly here, because of its direct applicability. In the constant-stress layer, on dimensional grounds, the turbulent transport coefficient in the y direction is $\epsilon \propto yu_*$. Using the Reynolds analogy one gets $\epsilon = \kappa u_* y$, where u_* is the shear velocity and κ is the Kármán constant. When the equilibrium state of particle transport in the y direction is reached, the downward flux due to gravity is balanced by the upward flux due to turbulent transport, that is

$$\epsilon \, dp/dy + wp = 0,$$

where p is the probability density of the particle position. Inserting the expression for ϵ into the above equation, and then integrating, gives

$$p = p_1 y^{-w/\kappa u_*}, \tag{1}$$

where p_1 is a constant. When $w/\kappa u_* < 1$,

$$\int_0^y p \, dy = \frac{p_1}{1 - w/\kappa u_*} y^{1-w/\kappa u_*} \tag{2}$$

and, when $w/\kappa u_* > 1$,

$$\int_y^\infty p\,dy = \frac{p_1}{w/\kappa u_* - 1} y^{1-w/\kappa u_*}. \tag{3}$$

Since

$$\int_0^y p\,dy$$

is proportional to the total number of particles between 0 and y, equation (2) shows that most of the particles in suspension reside above the constant-stress layer; this leads to the important conclusion that a particle stays in suspension almost all the time when $w/\kappa u_* < 1$. On the other hand, (3) shows that most of the particles in suspension reside in the constant-stress layer and that the total number of particles depends strongly on the conditions at the lower boundary of the flow, which, in our case, is the viscous sublayer, when $w/\kappa u_* > 1$.

Taking into consideration the argument given by Batchelor and the observations mentioned in the first paragraph of this section, it can be suggested that it is useful to consider two cases separately: (*a*) when the particle stays in suspension almost all the time and (*b*) when it does not.

Recent experiments by Corino & Brodkey (1969), Grass (1971) and others have shown that the wall region in turbulent flow consists of two zones each of which has its own structural character: (*a*) a viscous sublayer (or what Corino & Brodkey called the 'sublayer region') and (*b*) a generation region.

(*a*) *Viscous sublayer.* The flow in this region has a streaky character; very large lateral variation in the streamwise component of the velocity is correlated with the lateral velocity. This thin sublayer region is not of constant set thickness but rather is influenced by all events in the generation region. Corino & Brodkey† reported the viscous sublayer to be the region $0 \leqslant y^+ \leqslant 5$, of which the lowermost part $y^+ < 2{\cdot}5$ is essentially passive and the rest active. Here $y^+ \equiv yu_*/\nu$ and y is the distance from the wall.

(*b*) *Generation region.* Major generation and dissipation of turbulence occurs in this region. This region is the position of the majority of the so-called fluid ejection and fluid inrush phases. According to Corino & Brodkey the generation region is the zone $5 \leqslant y^+ \leqslant 70$. In the ejection phase low-velocity fluid is ejected away from the wall in the form of a three-dimensional disturbance occurring locally, and randomly with respect to time and longitudinal position. Ejected fluid originates from the lower zone of the generation region and has an instantaneous velocity component perpendicular to the wall which is as high as 30% of the longitudinal component. In the inrush phase high-velocity fluid penetrates towards the wall, again in the form of a three-dimensional disturbance.

Turning now to the case where the particle stays in suspension almost all the time, the necessary condition should be, from (2), $w/\kappa u_* < A$ in a general form, where A is a constant of order unity. Such a particle wandering close to the bottom during its travel over the cross-section may migrate downwards into the viscous sublayer owing to an inrush phase. Once a heavy particle is embedded in the viscous sublayer, particularly in the passive zone, it is hardly expected to be lifted into the body of the flow, since the experimental evidence reported by

† Corino & Brodkey made their observations in the region very near a pipe wall.

Corino & Brodkey showed that "there was often a connected movement of (fluid) particles (in the passive zone of the viscous sublayer) which occurred simultaneously with the ejection, but rarely did they possess sufficient... velocity to escape from the region". In that case, in order to maintain the particle in suspension and prevent it from leaving the main body of the flow by entering the viscous sublayer, another condition is that the particle size be greater than the thickness of the viscous sublayer, $d > \delta$. Then it can be said that a particle stays in suspension almost all the time if

$$\left.\begin{array}{l} w/\kappa u_* < A, \quad A = \text{constant of order unity,} \\ d > \delta. \end{array}\right\} \tag{4}$$

In the case where the particle does not stay in suspension all the time, the parameter $w/\kappa u_*$ is greater than A. In this case, as previously mentioned, most of the particles reside in the constant-stress layer, which overlaps considerably with the generation region. The lower boundary of the flow is the viscous sublayer; that is, the conditions here are the ones on which the total number of particles in suspension depends. In spite of the fact that the structure of turbulence in this region has been illuminated recently in some detail (Corino & Brodkey 1969; Grass 1971), the particle motion will remain an object of speculation until observations of the motion of heavy particles near the bottom under controlled conditions are made.

2.3. *Longitudinal dispersion when particles stay in suspension almost all the time*

Because of the presence of the free surface and the bottom, and the conditions $w/\kappa u_* < A$ and $d > \delta$, which imply that a particle cannot wander in the vertical direction and stays in suspension almost all the time, the velocity of the particle in the flow direction is necessarily a stationary random function of time as soon as the influence of the special choice of the point on the cross-section where the particle was released has been lost. It follows that the argument given by Batchelor & Townsend (1956, p. 360) should be applicable. Let $u(t)$ denote the fluctuation about the ensemble mean $\overline{u}$ of the component of the particle velocity in the flow direction. Following the same argument, it can be shown that $\overline{u}$ is equal to what might be called the particle discharge velocity, the measured rate of discharge of particles at some cross-section averaged over a long time divided by the product of the cross-sectional area and the cross-sectional average concentration of particles. The mean position of the particle is a distance

$$\overline{X}(t) = \overline{u}(t-t_0) \tag{5}$$

downstream from the point of release and the variance of the displacement about the mean is

$$\overline{X^2}(t) \simeq 2(t-t_0)\overline{u^2}\int_0^\infty S(t')\,dt' - 2\overline{u^2}\int_0^\infty t'\,S(t')\,dt' \quad \text{as} \quad t-t_0 \to \infty, \tag{6}$$

where $S(t')$ is the autocorrelation coefficient of the velocity u:

$$S(t') = \overline{u(t)\,u(t+t')}/\overline{u^2}. \tag{7}$$

The probability density function of the particle position in the x direction tends to a Gaussian form with a longitudinal diffusivity D_1 (longitudinal dispersion coefficient) as $t-t_0 \to \infty$, where

$$D_1 = \frac{1}{2}\frac{d\overline{X^2}}{dt} = \overline{u^2}\int_0^\infty S(t')\,dt'. \tag{8}$$

When the Reynolds number is large, one expects that the mean particle velocity $\bar{u}$ and the dispersion coefficient D_1 depend only on (a) the shear velocity u_* and the flow depth h because the velocity distribution across the channel is determined by these quantities (the viscous sublayer is not involved since $d > \delta$), and (b) the fall velocity w of the particle, if w is considered to represent both the inertial and gravity effects. On dimensional grounds, $\bar{u}$ and D_1 therefore should be given by

$$\frac{\bar{u}}{u_*} = f\left(\frac{w}{\kappa u_*}\right), \quad \frac{D_1}{hu_*} = g\left(\frac{w}{\kappa u_*}\right), \tag{9}$$

where κ, the Kármán constant, is employed here for convenience.

3. Prediction of mean velocity and longitudinal dispersion coefficient of heavy particles

3.1. *Formulation*

This section deals with the Eulerian formulation of the problem with the purpose of predicting the mean velocity and dispersion coefficient of the particles. For this, the initial particle distribution is considered to be in the form of a uniformly distributed plane source. From the considerations of §2.3, particles as a whole ultimately move with a certain mean velocity and spread out longitudinally. Conservation of mass gives the following equation:

$$\frac{\partial c}{\partial t} + U(y)\frac{\partial c}{\partial x} - w\frac{\partial c}{\partial y} = \frac{\partial}{\partial x}\left(\epsilon(y)\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(\epsilon(y)\frac{\partial c}{\partial y}\right). \tag{10}$$

The boundary conditions, which imply that there is no net transport across the boundaries, are

$$\epsilon(y)\,\partial c/\partial y + wc = 0 \quad \text{at} \quad y = 0, h \tag{11}$$

and the initial condition for a uniformly distributed instantaneous plane source is

$$c(x,y,0) = (m/h)\,\delta(x) \quad \text{for} \quad y \in [0,h] \quad \text{at} \quad t = 0. \tag{12}$$

Here c denotes the concentration of particles, m is the number of particles released per unit width and $\delta(x)$ is the Dirac delta function. As far as the application of (10) in the case where heavy particles are present is concerned, the experimental evidence shows that reasonable agreement between theory and experiment can be achieved in most cases by assuming that the downward flux of particles is equal to cw (provided that the particle fall velocities are in the Stokes range) and the turbulent transport coefficient of heavy particles is equal to that of fluid particles. Since this is outside the scope of the present study, further discussion is avoided.

To facilitate comparisons, the notation and non-dimensional parameters introduced by Aris (1956) and also used by Sayre (1968) will be employed here. The velocity U in the channel and the turbulent transport coefficient ϵ are written in the following forms:

$$U(y) = \overline{U}(1+\chi(y)), \quad \epsilon(y) = D\psi(y), \tag{13),(14}$$

where $\overline{U}$ and D are the cross-sectional averages of U and ϵ, respectively. Since particles ultimately move with a certain mean velocity U_s (it has been already mentioned in §2.3 that $U_s = \overline{u}$) it appears to be useful to write (10) with respect to axes moving with velocity U_s. Introducing the non-dimensional parameters

$$\left.\begin{aligned} \xi = (x-U_s t)/h, \quad \eta = y/h, \quad \tau = Dt/h^2, \quad C = c/(m/h^2), \\ \mu = \overline{U}h/D, \quad \mu_s = U_s h/D, \quad \nu_s = wh/D, \end{aligned}\right\} \tag{15}$$

the non-dimensional forms of (10)–(12) become

$$\frac{\partial C}{\partial \tau} + (\mu - \mu_s + \mu\chi)\frac{\partial C}{\partial \xi} - \nu_s \frac{\partial C}{\partial \eta} = \psi \frac{\partial^2 C}{\partial \xi^2} + \frac{\partial}{\partial \eta}\left(\psi \frac{\partial C}{\partial \eta}\right), \tag{16}$$

$$\psi\, \partial C/\partial \eta + \nu_s C = 0 \quad \text{at} \quad \eta = 0, 1, \tag{17}$$

$$C(\xi, \eta, 0) = \delta(\xi) \quad \text{for} \quad \eta \in [0,1] \quad \text{at} \quad \tau = 0. \tag{18}$$

Assuming that the velocity distribution is logarithmic and employing the Reynolds analogy, $\mu\chi$ and ψ are given by

$$\mu\chi = 6\kappa^{-2}(1+\log\eta), \quad \psi = 6\eta(1-\eta) \tag{19),(20}$$

and D in (14) becomes $\frac{1}{6}\kappa h u_*$. Then the fall-velocity parameter ν_s becomes

$$\nu_s = 6w/\kappa u_* = 6\beta,$$

where $\beta \equiv w/\kappa u_*$ and will be used throughout the study as the characteristic parameter representing the effect of gravity. If the pth moment of the concentration is defined as

$$C_p = \int_{-\infty}^{\infty} \xi^p C\, d\xi$$

equations (16)–(18) become (Aris 1956)

$$\frac{\partial C_p}{\partial \tau} - (\mu - \mu_s + \mu\chi)\, pC_{p-1} - \nu_s \frac{\partial C_p}{\partial \eta} = p(p-1)\,\psi C_{p-2} + \frac{\partial}{\partial \eta}\left(\psi \frac{\partial C_p}{\partial \eta}\right), \tag{21}$$

$$\psi\, \partial C_p/\partial \eta + \nu_s C_p = 0 \quad \text{at} \quad \eta = 0, 1, \tag{22}$$

$$C_p(\eta, 0) = \left\{\begin{matrix} 1 & \text{for} & p = 0 \\ 0 & \text{for} & p > 0 \end{matrix}\right\} \quad \text{at} \quad \tau = 0. \tag{23}$$

If m_p is defined as the cross-sectional average of C_p,

$$m_p = \int_0^1 C_p\, d\eta,$$

then further transformation of (21) and (23) gives

$$\frac{dm_p}{d\tau}-p\int_0^1(\mu-\mu_s+\mu\chi)\,C_{p-1}\,d\eta-p(p-1)\int_0^1\psi C_{p-2}\,d\eta=0, \tag{24}$$

$$m_p(0)=\begin{Bmatrix}1 & \text{for} & p=0\\ 0 & \text{for} & p>0\end{Bmatrix}\quad\text{at}\quad\tau=0. \tag{25}$$

The transformations applied here are exactly the same as those first introduced by Aris (1956) and also employed by Sayre (1968).

3.2. *Zeroth moment of concentration*

From (24), for $p = 0$, $dm_0/d\tau = 0$ and, using $m_0(0) = 1$ [equation (25)], m_0 is found to be equal to unity at any time τ. Equations (21)–(23) for $p = 0$, together with $m_0 = 1$, are employed to find the solution for $C_0(\eta, \tau)$, that is,

$$\frac{\partial C_0}{\partial\tau}-6\beta\frac{\partial C_0}{\partial\eta}=\frac{\partial}{\partial\eta}\left(6\eta(1-\eta)\frac{\partial C_0}{\partial\eta}\right), \tag{26}$$

$$\eta(1-\eta)\,\partial C_0/\partial\eta+\beta C_0=0\quad\text{at}\quad\eta=0,1, \tag{27}$$

$$C_0(\eta,0)=1\quad\text{at}\quad\tau=0, \tag{28}$$

$$m_0=\int_0^1 C_0(\eta,\tau)\,d\eta=1. \tag{29}$$

$C_0(\eta,\tau)$ is found to be as follows (see appendix A):

$$C_0=\frac{\sin\pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^{\beta}+\sum_{K=1}^{\infty}a_K\left(\frac{1-\eta}{\eta}\right)^{\beta}F(-K,1+K;1-\beta;\eta)\exp\{-6(K^2+K)\tau\}$$
$$\text{for}\quad\beta\neq0,\quad\beta<1, \tag{30}$$

where the constants a_K are given by equation (A 12). In the case of neutrally buoyant particles, $\beta = 0$, since (*a*) the initial condition (28) is $C_0(\eta, 0) = 1$ and (*b*) $C_0(\eta,\tau)$ should tend to unity as $\tau\to\infty$ because there is no gravitational effect in this case, then $C_0(\eta, \tau) = 1$ for all values of time.

Writing the hypergeometric series in (30) in terms of Jacobi polynomials [equation (A 11)] and making use of tables of coefficients for the Jacobi polynomials (Abramowitz & Stegun 1968, p. 793), $C_0(\eta,\tau)$ was evaluated for $\beta = 0{\cdot}1$ and $0{\cdot}3$ (figure 2). It was found that C_0 reaches the state of equilibrium at approximately $\tau = 0{\cdot}5$, after which it becomes a time-independent function and can be expressed as

$$C_0(\eta,\tau)=\frac{\sin\pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^{\beta},\quad\tau\geqslant0{\cdot}5. \tag{31}$$

Sayre (1968, p. 31) arrived at the same result by his numerical solution.

3.3. *Mean particle velocity*

Equations (24) and (25) with $p = 1$ are employed to predict the mean particle velocity:

$$\frac{dm_1}{d\tau}-\int_0^1(\mu-\mu_s+\mu\chi)\,C_0(\eta,\tau)\,d\eta=0, \tag{32}$$

$$m_1(0)=0\quad\text{at}\quad\tau=0, \tag{33}$$

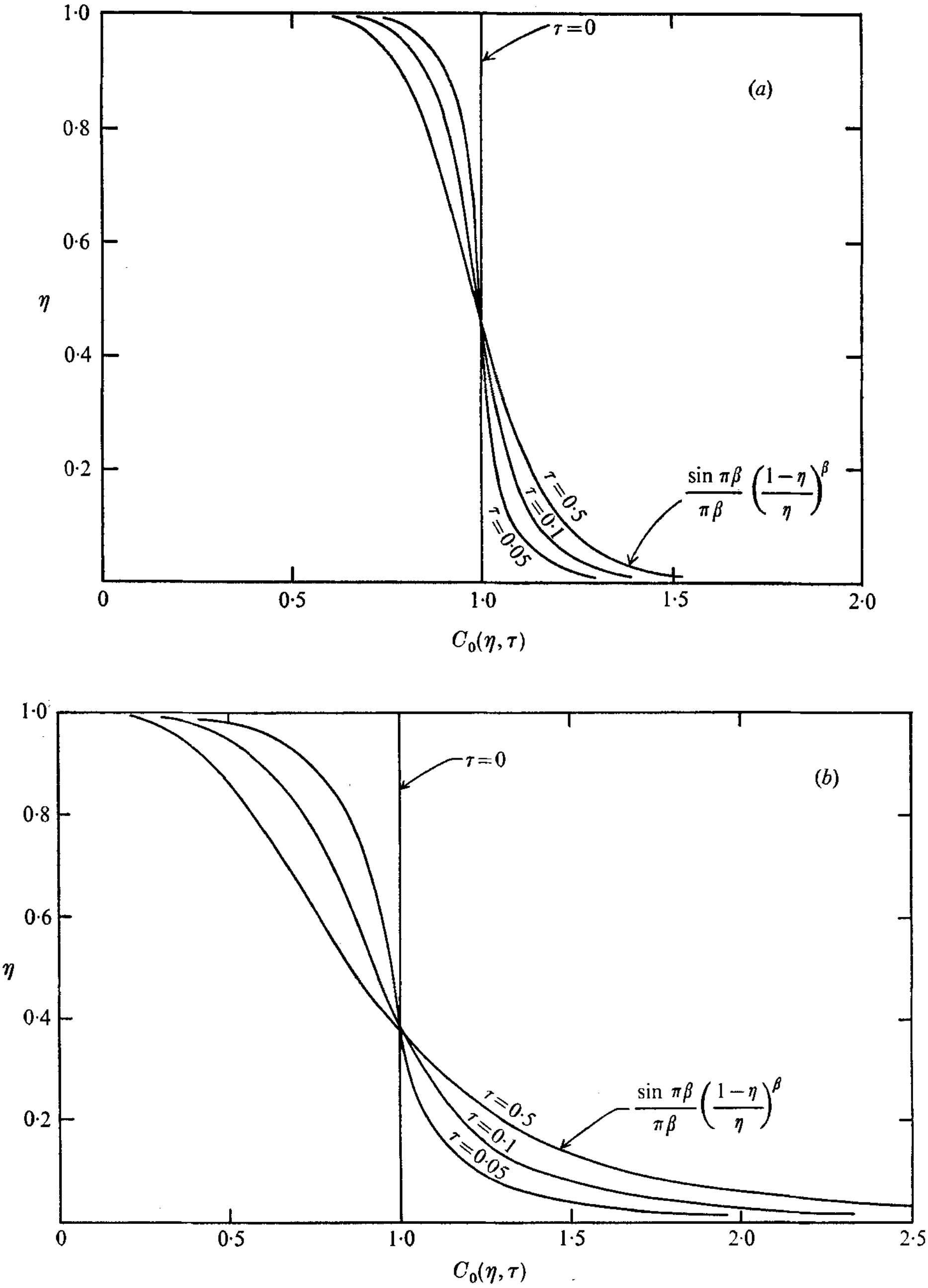

FIGURE 2. $C_0(\eta, \tau)$ for (*a*) $\beta = w/\kappa u_* = 0{\cdot}1$ and (*b*) $\beta = w/\kappa u_* = 0{\cdot}3$.

of which the latter implies that the origin of ξ is chosen in the original plane of the centre of gravity. $dm_1/d\tau$ is the rate of the mean particle displacement relative to the ξ axis moving with the ultimate mean particle velocity U_s (or in non-dimensional form μ_s). Thus

$$dm_1/d\tau \to 0 \quad \text{as} \quad \tau \to \infty$$

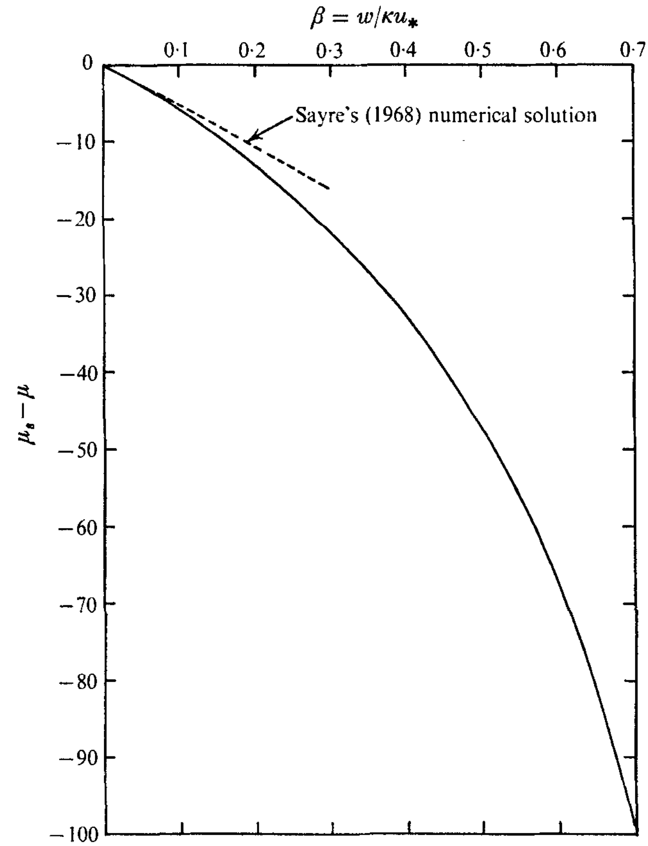

FIGURE 3. Mean velocity of heavy particles relative to the mean flow velocity *vs.* the fall-velocity parameter $\beta = w/\kappa u_*$.

and from (32), using (30), one gets

$$\mu_s - \mu = \int_0^1 \mu\chi \lim_{\tau\to\infty} C_0(\eta,\tau)\, d\eta = \int_0^1 \mu\chi \frac{\sin \pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^\beta d\eta, \qquad (34)$$

which gives the mean particle velocity relative to the mean flow velocity. This expression is the same as that used by Binnie & Phillips (1958) and Barnard & Binnie (1963) to predict the mean particle velocity in a pipe flow.

Integration is carried out in terms of beta and psi functions. By writing the beta function in terms of gamma functions and after some algebra, $\mu_s - \mu$ is obtained in the form

$$\mu_s - \mu = 6\kappa^{-2}[1 + \Psi(1-\beta) - \Psi(2)], \quad \beta < 1, \qquad (35)$$

where Ψ is the psi function. Using the tables of Abramowitz & Stegun (1968, p. 267) $\mu_s - \mu$ has been plotted against β in figure 3. In the calculation κ was taken as 0·42. For comparison, Sayre's numerical solution (1968, figures 3–15) has been plotted too.

Inserting (30) into (32) and using (34) and (A 8), equation (32) becomes

$$\frac{dm_1}{d\tau} = \sum_{K=1}^{\infty} a_K \left\{\int_0^1 \mu\chi \left(\frac{1-\eta}{\eta}\right)^\beta F(-K, 1+K; 1-\beta; \eta)\, d\eta\right\} \exp\{-6(K^2+K)\tau\}.$$

From the above equation, using the initial condition (33), one has

$$m_1(\tau) = \sum_{K=1}^{\infty} \frac{a_K}{6(K^2+K)} \int_0^1 \mu\chi \left(\frac{1-\eta}{\eta}\right)^{\beta} F(-K, 1+K; 1-\beta; \eta)\, d\eta$$
$$+ \sum_{K=1}^{\infty} - \frac{a_K}{6(K^2+K)} \left\{ \int_0^1 \mu\chi \left(\frac{1-\eta}{\eta}\right)^{\beta} F(-K, 1+K; 1-\beta; \eta)\, d\eta \right\}$$
$$\times \exp\{-6(K^2+K)\tau\}. \tag{36}$$

Thus the mean particle position ultimately moves to

$$m_1(\infty) = \sum_{K=1}^{\infty} \frac{a_K}{6(K^2+K)} \int_0^1 \mu\chi \left(\frac{1-\eta}{\eta}\right)^{\beta} F(-K, 1+K; 1-\beta; \eta)\, d\eta.$$

3.4. *First moment of concentration*

The equations for the first moment of the concentration, (21)–(23) for $p = 1$, are

$$\frac{\partial C_1}{\partial \tau} - 6\beta \frac{\partial C_1}{\partial \eta} - \frac{\partial}{\partial \eta}\left(6\eta(1-\eta)\frac{\partial C_1}{\partial \eta}\right) = (\mu - \mu_s + \mu\chi)\, C_0, \tag{37}$$

$$\eta(1-\eta)\, \partial C_1/\partial \eta + \beta C_1 = 0 \quad \text{at} \quad \eta = 0, 1, \tag{38}$$

$$C_1(\eta, 0) = 0 \quad \text{at} \quad \tau = 0. \tag{39}$$

Equation (36) will also be employed. $C_1(\eta, \tau)$ is obtained in the following form (see appendix B):

$$C_1 = \phi(\eta) + \sum_{K=1}^{\infty} a_K f_K(\eta) \exp\{-6(K^2+K)\tau\}$$
$$+ \sum_{K=1}^{\infty} d_K \left(\frac{1-\eta}{\eta}\right)^{\beta} F(-K, 1+K; 1-\beta; \eta) \exp\{-6(K^2+K)\tau\}, \tag{40}$$

where

$$\phi = A_\phi \left(\frac{1-\eta}{\eta}\right)^{\beta} - \frac{\sin \pi\beta}{\pi\beta} \left(\frac{1-\eta}{\eta}\right)^{\beta} \int_0^{\eta} \left(\frac{1-\eta}{\eta}\right)^{-\beta} \frac{1}{6\eta(1-\eta)}\, d\eta$$
$$\times \int_0^{\eta} (\mu - \mu_s + \mu\chi) \left(\frac{1-\eta}{\eta}\right)^{\beta} d\eta, \tag{41}$$

in which A_ϕ is a constant. It should be noticed that $C_1 \to \phi$ as $\tau \to \infty$. As far as the solution for ϕ is concerned, the only condition on β is $\beta < 1$. Therefore, for neutrally buoyant particles, one has

$$\phi = A_\phi - \int_0^{\eta} \frac{1}{6\eta(1-\eta)}\, d\eta \int_0^{\eta} \mu\chi\, d\eta \quad \text{for} \quad \beta = 0. \tag{42}$$

3.5. *Longitudinal dispersion coefficient of particles*

The longitudinal dispersion coefficient defined in (8) can be written in non-dimensional form as

$$D_1/hu_* = \tfrac{1}{6}\kappa \lim_{\tau \to \infty} \tfrac{1}{2} dm_2/d\tau. \tag{43}$$

From (24)

$$\frac{1}{2}\frac{dm_2}{d\tau} = 6\int_0^1 \eta(1-\eta)\, C_0\, d\eta + \int_0^1 (\mu - \mu_s + \mu\chi)\, C_1\, d\eta. \tag{44}$$

$\beta = w/\kappa u_*$	D_1/hu_*
0·1	6·80
0·2	8·41
0·3	10·53
0·4	13·39
0·5	17·42
0·6	23·02

TABLE 1

Substituting the ultimate distributions of C_0 and C_1 as $\tau \to \infty$ [(31) and (41), respectively] into the above equation, one gets

$$\frac{D_1}{hu_*} = \frac{\kappa}{6}(1-\beta^2) - \frac{\kappa}{36}\frac{\sin\pi\beta}{\pi\beta}\int_0^1 (\mu-\mu_s+\mu\chi)\left(\frac{1-\eta}{\eta}\right)^{\beta} d\eta$$
$$\times \int_0^{\eta}\left(\frac{1-\eta}{\eta}\right)^{-\beta}\frac{1}{\eta(1-\eta)}d\eta \int_0^{\eta}(\mu-\mu_s+\mu\chi)\left(\frac{1-\eta}{\eta}\right)^{\beta} d\eta, \quad \beta < 1. \quad (45)$$

The first term on the right-hand side is the contribution of the longitudinal turbulent diffusion and, as expected, is negligible in comparison with the second. The second, which is the main portion of the expression, arises because of the combined action of the velocity gradient and the vertical diffusion of particles under gravity. For neutrally boyant particles, $\beta = 0$, equation (45) reduces to that obtained by Elder (1959).

The expression (45) was computed numerically with the aid of a digital computer for $\beta = 0·1, 0·2, \ldots, 0·6$. In the numerical integration the usual trapezoidal rule was used because of its convenience for this particular case. The interval [0, 1] was divided into small intervals. Two different interval lengths, one in the neighbourhoods of $\eta = 0$ and $\eta = 1$, and the other between those two, were used. The integration was performed at each step by choosing the interval sizes smaller than those of the previous ones. This procedure was repeated until the difference between two subsequent outputs was judged to be small enough. In the calculations κ was taken to be 0·42. The results are given in table 1. For comparison, the longitudinal dispersion coefficient has been plotted in the form $D_1(\beta)/D_1(0)$ together with Sayre's numerical solution in figure 4, where $D_1(0)$ is taken as 5·52, which is the converted form of Elder's (1959) prediction for $\kappa = 0·42$ (5·93 for $\kappa = 0·41$). The results show that the dispersion coefficient increases with the fall velocity of particles. This behaviour can be interpreted as follows. Equation (31) implies that a heavier particle spends most of the time very close to the bottom, where the velocity gradient is greater than in any other region. A lighter particle, however, has less chance of travelling near the bottom, but more chance far from the bottom, where the velocity gradient is small. On the other hand, the greater the velocity gradient, the larger the longitudinal dispersion. A heavier particle, therefore, should have a larger dispersion coefficient.

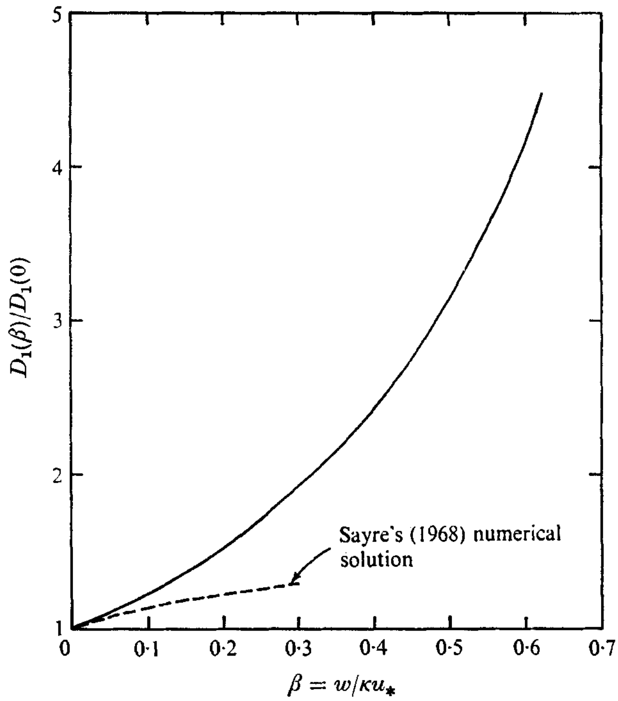

FIGURE 4. Ratio of longitudinal dispersion coefficient of heavy particles to that of neutrally buoyant particles *vs.* the fall-velocity parameter $\beta = w/\kappa u_*$.

3.6. *Discussion*

As has been seen, Sayre's (1968) numerical solutions for the mean velocity and the dispersion coefficient of particles differ significantly from the analytical results given in the present study, particularly for the large values of β; the discrepancy in $\mu_s - \mu$ is 12 % for $\beta = 0{\cdot}1$ and 37 % for $\beta = 0{\cdot}3$, and the discrepancy in $D_1(\beta)/D_1(0)$ is 7 % for $\beta = 0{\cdot}1$ and 32 % for $\beta = 0{\cdot}3$. The author believes that the discrepancies are due to the large depth increment $\Delta\eta = 0{\cdot}05$ which Sayre (1968) used in his numerical solution. This, obviously, moderates the effect of $\mu\chi \to -\infty$ and $C_0 \to +\infty$ at $\eta = 0$. By reducing $\Delta\eta$ to 0·01, using Simpson's rule of numerical integration and assuming that the values of $\mu\chi$ and C_0 are the same at $\eta = 0$ as at $\eta = 0{\cdot}001$, Sayre (1973, private communication) pointed out that he obtained a value of $\mu_s - \mu = -21{\cdot}2$ for $\beta = 0{\cdot}3$, which is very close to the value of $-21{\cdot}9$ in the present study. Also pointing out that approximately 50 % of the contribution to $\mu_s - \mu$ comes from the bottom 2 % of the flow field, i.e. $0 \leqslant \eta \leqslant 0{\cdot}02$ (which shows a tremendous sensitivity of the result to values of the velocity and concentration very close to the bottom), Sayre confirmed the fact that the discrepancies are due to the large depth increment. This sensitivity is even more pronounced for $D_1(\beta)/D_1(0)$, which involves second-order terms. As can be seen from figures 3 and 4, the discrepancies decrease as β decreases; the discrepancy is the least for $\beta = 0$. This is because the large depth increment in the numerical solution becomes less pronounced in moderating the effect of C_0 near $\eta = 0$ when C_0 goes to a weak infinity at $\eta = 0$ for a smaller value of β. This, too, implicitly indicates the moderating effect of the large depth increment used in the numerical model and the sensitivity of the outcome to velocities and concentrations near the bottom.

4. Conclusions

Making use of Batchelor's (1965) argument for particle transport in the constant-stress layer, two types of particle motion may be considered separately; one is when the particle stays in suspension almost all the time and the other when it does not. In the first case, in addition to the necessary condition $w/\kappa u_* < A$, it appears that to prevent the particle from leaving the flow the particle size should be greater than the thickness of the viscous sublayer. At present very little is known about the case where the particle does not stay in suspension all the time and observations of the motion of heavy particles close to the bottom under controlled conditions are needed.

This paper essentially deals with the case where the particle stays in suspension almost all the time. The problem, in this case, is formulated in the Eulerian sense and then the Aris moment transformations are applied. The zeroth and first moments of the concentration, the mean velocity and longitudinal dispersion coefficient of particles are determined analytically.

(*a*) The zeroth moment C_0 of the concentration is the probability density function of the projection on a cross-sectional plane of the particle position. It is found that for the initial condition of a uniformly distributed instantaneous plane source, $C_0(\eta, \tau)$ tends to the expression obtained by employing the balance between the downward flux due to gravity and the upward flux due to turbulent transport, when $\tau \geqslant 0{\cdot}5$.

(*b*) It is also found that the mean particle velocity decreases and the longitudinal dispersion coefficient of particles increases with the fall velocity. In fact, a heavier particle spends most of the time very close to the bottom, where the velocity is lower and the velocity gradient is greater than any other region. This implies that a heavier particle should have a smaller mean velocity and a larger dispersion coefficient.

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Appendix A. The solution for $C_0(\eta,\tau)$

With the transformation $T = 6\tau$, equation (26) may be written as

$$\frac{\partial C_0}{\partial T} - \beta\frac{\partial C_0}{\partial \eta} = \frac{\partial}{\partial \eta}\left(\eta(1-\eta)\frac{\partial C_0}{\partial \eta}\right).$$

On separating the variables according to $C_0(\eta, T) = P(\eta)\,G(T)$, the above equation becomes

$$\frac{dG}{dT}\frac{1}{G} = \left[\beta\frac{dP}{d\eta} + \frac{d}{d\eta}\left(\eta(1-\eta)\frac{dp}{d\eta}\right)\right]\frac{1}{P} = -\alpha^2 + \tfrac{1}{4}. \tag{A 1}$$

The factor $\frac{1}{4}$ is added for convenience. The time-dependent part is found to be

$$G = G_0 \exp\{-(\alpha^2 - \tfrac{1}{4})\,T\}, \quad G_0 = \text{constant}. \tag{A 2}$$

The function P satisfies the hypergeometric differential equation

$$\eta(1-\eta)\,P'' + (1+\beta-2\eta)\,P' + (\alpha^2 - \tfrac{1}{4})\,P = 0. \tag{A 3}$$

The coefficients a, b and c if the hypergeometric differential equation is written in the form $z(1-z)\,u'' + [c-(a+b+1)\,z]\,u' - abu = 0$ are, in our case, as follows:

$$a = \tfrac{1}{2} + \alpha, \quad b = \tfrac{1}{2} - \alpha, \quad c = 1 + \beta.$$

From Erdélyi *et al.* (1953, pp. 71 and 105) the solution satisfying the conditions that either $a = \frac{1}{2} + \alpha$ or $b = \frac{1}{2} - \alpha$ is a negative integer $-m$ and the other is a positive integer $1 + l$, where m and l denote non-negative integers, and neither $c = 1 + \beta$ nor $c - a - b = \beta$ is an integer is

$$P = AF(\tfrac{1}{2}+\alpha, \tfrac{1}{2}-\alpha; 1+\beta; \eta) + B\left(\frac{1-\eta}{\eta}\right)^{\beta} F(\tfrac{1}{2}-\alpha, \tfrac{1}{2}+\alpha; 1-\beta; \eta), \tag{A 4}$$

provided that $\beta \neq 0$. Here A and B are constants and F is a Gaussian hypergeometric series which, in the present case, reduces to a polynomial in η of degree n since either $\frac{1}{2}+\alpha$ or $\frac{1}{2}-\alpha$ is a negative integer ($n = 0, -1, -2, \ldots$). Application of the boundary condition (27) at $\eta = 0$ gives $A = 0$. Then, from (A 2) and (A 4), one gets

$$C_0 = G_0 B\left[\left(\frac{1-\eta}{\eta}\right)\right]^{\beta} F(\tfrac{1}{2}-\alpha, \tfrac{1}{2}+\alpha; 1-\beta; \eta)\exp\{-(\alpha^2 - \tfrac{1}{4})\,T\}. \tag{A 5}$$

C_0, actually, is equal to the probability density function of the projection on a cross-sectional plane of the particle position at any instant T since, as is known, there exists a relation between concentration and the probability density function of the particle position; $c(x,y,t) = mp(x,y,t)$. C_0, therefore, must be expected to reach a state of equilibrium after which it no longer depends on time. The expression (A 5) becomes a time-independent function when $\alpha = \frac{1}{2}$:

$$C_0 = G_0 B\left(\frac{1-\eta}{\eta}\right)^{\beta} F(0, 1; 1-\beta; \eta) = G_0 B\left(\frac{1-\eta}{\eta}\right)^{\beta}.$$

In the case of equilibrium, by determining the constant $G_0 B$ by application of (29), C_0 is found to be

$$C_0 = \frac{\sin \pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^{\beta}, \quad \beta^2 < 1, \tag{A 6}$$

which is the same expression as that obtained by Elder (1959), and first introduced in a slightly different form by Rouse (1937). For the non-equilibrium case it is assumed that

$$C_0 = \frac{\sin \pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^{\beta} + \sum_{K=1}^{\infty} a_K \left(\frac{1-\eta}{\eta}\right)^{\beta} F(\tfrac{1}{2}-\alpha_K, \tfrac{1}{2}+\alpha_K; 1-\beta; \eta)\exp\{-(\alpha_K^2 - \tfrac{1}{4})\,T\}, \tag{A 7}$$

where a_K is a constant. In order that $C_0 \to C_0(\eta)$ as $T \to \infty$, the constant α should exceed $\frac{1}{2}$. To obtain the sequence of constants α_K ($K = 1, 2, \ldots$), equation (29) is employed. This gives

$$\int_0^1 \left(\frac{1-\eta}{\eta}\right)^\beta F(\tfrac{1}{2}-\alpha_K, \tfrac{1}{2}+\alpha_K; 1-\beta; \eta)\, d\eta = 0 \tag{A8}$$

and evaluating the integral one has

$$\Gamma(1-\beta)\,\Gamma(1+\beta)/\Gamma(\alpha+\tfrac{3}{2})\,\Gamma(\tfrac{3}{2}-\alpha) = 0,$$

where the integration brings in the condition $\beta < 1$. Keeping in mind that $\alpha > \frac{1}{2}$, the positive zeros of the above equation are the same as the poles of the gamma function $\Gamma(\frac{3}{2}-\alpha)$, so that the sequence of constants α is

$$\alpha_K = K+\tfrac{1}{2}, \quad K = 1, 2, 3, \ldots. \tag{A9}$$

Then (A 7) becomes

$$C_0 = \frac{\sin \pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^\beta + \sum_{K=1}^{\infty} a_K \left(\frac{1-\eta}{\eta}\right)^\beta F(-K, 1+K; 1-\beta; \eta) \exp\{-(K^2+K)T\}. \tag{A10}$$

C_0 can also be written in terms of Jacobi polynomials (Abramowitz & Stegun 1968, p. 561):

$$C_0 = \frac{\sin \pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^\beta + \sum_{K=1}^{\infty} a_K \frac{K!}{(1-\beta)_K}\left(\frac{1-\eta}{\eta}\right)^\beta P_K^{(-\beta,\beta)}(1-2\eta)\exp\{-(K^2+K)\,T\}, \tag{A11}$$

where the $(1-\beta)_K$ are Pochammer's symbols.

The constants a_K are determined by employing the initial condition (28), which gives

$$1 = \frac{\sin \pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^\beta + \sum_{K=1}^{\infty} a_K \frac{K!}{(1-\beta)_K}\left(\frac{1-\eta}{\eta}\right)^\beta P_K^{(-\beta,\beta)}(1-2\eta).$$

The Jacobi polynomials $G_K(p, q, \eta)$ are orthogonal on the interval $0 \leqslant \eta \leqslant 1$ with respect to the weighting function $(1-\eta)^{p-q}\eta^{q-1}$, which, in this case, equals $[(1-\eta)/\eta]^\beta$ (Abramowitz & Stegun 1968, p. 773). Thus, by writing the above equation in terms of G_K,

$$1 = \frac{\sin \pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^\beta + \sum_{K=1}^{\infty} a_K \frac{\Gamma(2K+1)}{(1-\beta)_K\,\Gamma(K+1)}\left(\frac{1-\eta}{\eta}\right)^\beta G_K(1, 1+\beta, 1-\eta),$$

and multiplying the latter by $G_M(1, 1+\beta, 1-\eta)$, setting $K = M$ and taking into account that $\beta < 1$, the constant a_K is obtained as

$$a_K = \frac{(2K+1)\,\Gamma(K+1-\beta)}{\Gamma^2(1-\beta)\,\Gamma(K+1+\beta)}\,{}_3F_2(-K, K+1, 1; 1-\beta, 2; 1), \tag{A12}$$

where ${}_3F_2$ is a generalized hypergeometric series (Gradshteyn & Ryzhik 1965, p. 1045). The boundary condition (27) at $\eta = 1$ is automatically satisfied.

Appendix B. The solution for $C_1(\eta,\tau)$

The solution of (37) consists of two parts: one is the particular solution and the other the complementary function. One can write the particular integral C_1^P as

$$C_1^P = \phi(\eta) + \sum_{K=1}^{\infty} a_K f(\eta)\exp\{-6(K^2+K)\tau\}, \tag{B 1}$$

where $\phi(\eta)$ arises from the part of C_0 which is a function only of η. Substituting this expression into (37), it is found that ϕ and f should satisfy the following equations:

$$\frac{d}{d\eta}\left(\eta(1-\eta)\frac{d\phi}{d\eta}\right)+\beta\frac{d\phi}{d\eta} = -\tfrac{1}{6}(\mu-\mu_s+\mu\chi)\frac{\sin\pi\beta}{\pi\beta}\left(\frac{1-\eta}{\eta}\right)^{\beta}, \tag{B 2}$$

$$\frac{d}{d\eta}\left(\eta(1-\eta)\frac{df}{d\eta}\right)+\beta\frac{df}{d\eta}+(K^2+K)f = -\tfrac{1}{6}(\mu-\mu_s+\mu\chi)\times\left(\frac{1-\eta}{\eta}\right)^{\beta}F(-K,1+K;1-\beta;\eta). \tag{B 3}$$

The latter implies that f is to be determined for each K.

The complementary function is the solution of the homogeneous equation, which is identical to that for C_0. In the same way as in appendix A the complementary function C_1^C can be written as

$$C_1^C = \sum_{K=1}^{\infty}\left\{e_K F\left(\tfrac{1}{2}+\lambda_K,\tfrac{1}{2}-\lambda_K;1+\beta;\eta\right)+d_K\left(\frac{1-\eta}{\eta}\right)^{\beta}\times F\left(\tfrac{1}{2}-\lambda_K,\tfrac{1}{2}+\lambda_K;1-\beta;\eta\right)\right\}\exp\{-6(\lambda_K^2-\tfrac{1}{4})\tau\}, \tag{B 4}$$

where e_K and d_K are constants.

Inserting the complete solution $C_1^p+C_1^C$ into the boundary condition (38) at $\eta = 0$ one gets

$$e_K = 0,$$

$$\left.\begin{aligned}\eta(1-\eta)\frac{d\phi}{d\eta}+\beta\phi &= 0 \quad &(\text{B }5)\\ \eta(1-\eta)\frac{df_K}{d\eta}+\beta f_K &= 0 \quad &(\text{B }6)\end{aligned}\right\}\quad \text{at}\quad \eta = 0$$

and, again inserting the complete solution into (36) in the form

$$\int_0^1 C_1(\eta,\tau)\,d\eta,$$

the following equations are obtained:

$$\int_0^1 \phi(\eta)\,d\eta = \sum_{K=1}^{\infty}\frac{a_K}{6(K^2+K)}\int_0^1 \mu\chi\left(\frac{1-\eta}{\eta}\right)^{\beta}F(-K,1+K;1-\beta;\eta)\,d\eta, \tag{B 7}$$

$$\int_0^1 f_K(\eta)\,d\eta = -\frac{1}{6(K^2+K)}\int_0^1 \mu\chi\left(\frac{1-\eta}{\eta}\right)^{\beta}F(-K,1+K;1-\beta;\eta)\,d\eta, \tag{B 8}$$

$$\int_0^1\left(\frac{1-\eta}{\eta}\right)^{\beta}F(\tfrac{1}{2}-\lambda_K,\tfrac{1}{2}+\lambda_K;1-\beta;\eta)\,d\eta = 0. \tag{B 9}$$

From (B 9) λ is found to be identical to α: $\lambda_K \equiv \alpha_K$. Then the complete solution is written as

$$C_1 = \phi(\eta) + \sum_{K=1}^{\infty} a_K f_K(\eta) \exp\{-6(K^2+K)\tau\} + \sum_{K=1}^{\infty} d_K \left(\frac{1-\eta}{\eta}\right)^{\beta} F(-K, 1+K, 1-\beta; \eta) \exp\{-6(K^2+K)\tau\}. \quad \text{(B 10)}$$

The function ϕ now can be determined by (B 2), (B 5) and (B 7). Integration of (B 2) and then application of the boundary condition (B 5) give

$$\phi' = -\frac{\sin \pi\beta}{\pi\beta} \frac{1}{6\eta(1-\eta)} \int_0^{\eta} (\mu - \mu_s + \mu\chi) \left(\frac{1-\eta}{\eta}\right)^{\beta} d\eta - \frac{\beta}{\eta(1-\eta)} \phi. \quad \text{(B 11)}$$

Solution of the above first-order differential equation gives (Murphy 1960, p. 13)

$$\phi = A_{\phi} \left(\frac{1-\eta}{\eta}\right)^{\beta} - \frac{\sin \pi\beta}{\pi\beta} \left(\frac{1-\eta}{\eta}\right)^{\beta} \int_0^{\eta} \left(\frac{1-\eta}{\eta}\right)^{-\beta} \frac{1}{6\eta(1-\eta)} d\eta \times \int_0^{\eta} (\mu - \mu_s + \mu\chi) \left(\frac{1-\eta}{\eta}\right)^{\beta} d\eta, \quad \text{(B 12)}$$

where A_{ϕ} is a constant and is chosen to satisfy (B 7). The boundary condition (38) at $\eta = 1$ and the initial condition (39) have not been used so far. The constants d_K in (B 10) are chosen to satisfy (39). On the other hand, provided that $f_K(\eta)$ is to be determined such that the boundary condition at $\eta = 1$ is satisfied, it can be easily seen that the boundary condition at $\eta = 1$ is automatically satisfied by the rest of the complete solution for C_1.

The analysis for the solution for C_1 is performed here with the purpose of predicting the longitudinal dispersion coefficient. As can be seen, one only needs to determine ϕ. Even the constant A_{ϕ} is not needed for this purpose because it drops out of the expression for the dispersion coefficient [see (43) and (44)] when ϕ is inserted.

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